Final Exam Review Sheet
MATH 283

The final exam is comprehensive and will contain material from throughout the course, though there will be a preference towards newer material on which you haven't been tested yet. This review sheet only includes the material that we've covered since the last exam; please refer to the Exam I, II, and III review sheets for the earlier material. Again, it is important to note that this review sheet is not necessarily exhaustive. You are responsible for the material presented in class, homework, and quizzes too.

- Chapter 16: Vector Calculus
  - §16.1 Vector Fields
  - §16.2 Line Integrals
  - §16.3 The Fundamental Theorem for Line Integrals
  - §16.4 Green's Theorem
  - §16.5 Curl and Divergence
  - §16.6 Parametric Surfaces and Surface Area
  - §16.7 Surface Integrals
  - §16.8 Stokes' Theorem
  - §16.9 The Divergence Theorem

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1. Sketch the vector field.
   (a) $\vec{F}(x,y) = \frac{1}{2}x\ \hat{\mathbf{i}} + y\ \hat{\mathbf{j}}$
   (b) $\vec{F}(x,y) = \dfrac{y\ \hat{\mathbf{i}} - x\ \hat{\mathbf{j}}}{\sqrt{x^2+y^2}}$
2. Evaluate the line integral.
   (a) $\int_C xy^4\ ds$, where C is the right half of the circle $x^2+y^2=16$.
   (b) $\int_C xe^y\ ds$, where C is the line segment from $(2,0)$, to $(5,4)$.
   (c) $\int_C y^2 z\ ds$, where C is the line segment from $(3,1,2)$ to $(1,2,5)$.
   (d) $\int_C x^2\ dx + y^2\ dy$, where C consists of the arc of the circle $x^2+y^2=4$ from $(2,0)$ to $(0,2)$ followed by the line segment from $(0,2)$ to $(4,3)$.
3. Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where C is the curve given by the vector function $\vec{r}(t)$.
   (a) $\vec{F}(x,y) = xy^2\ \hat{\mathbf{i}} - x^2\ \hat{\mathbf{j}}$;
   $\vec{r}(t) = t^3\ \hat{\mathbf{i}} + t^2\ \hat{\mathbf{j}}, \quad 0 \le t \le 1$.

(b) $\vec{F}(x,y,z) = x\,\hat{\imath} + y\,\hat{\jmath} + xy\,\hat{\boldsymbol{k}}$;
$\vec{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t\,\hat{\boldsymbol{k}}, \quad 0 \le t \le \pi$.

4. Let $\vec{F}(x,y) = -y\,\hat{\imath} + x\,\hat{\jmath}$ and let C be the line segment from $(2,1)$ to $(5,4)$.
Note: $\vec{F}$ is not conservative since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ and so you cannot use the fundamental theorem of line integrals to solve parts (b) and (d).

   (a) Find a parameterization $\vec{r}(t) = \langle x(t), y(t)\rangle$ for the line segment C such that $0 \le t \le 1$.

   (b) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ using the parameterization given in (a).

   (c) Find a parameterization $\vec{r}(t) = \langle x(t), y(t)\rangle$ for the line segment C such that $0 \le t \le \frac{1}{2}$.

   (d) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ using the parameterization given in (c).

5. Let $\vec{F}(x,y) = y\,\hat{\imath} + x\,\hat{\jmath}$ and let C be the line segment from $(2,1)$ to $(5,4)$.

   (a) Verify that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Note that the partial derivatives the component functions of $\vec{F}$ are defined and continuous on the entire xy-plane. Thus $\vec{F}$ is conservative and so there exists a potential function $f(x,y)$ such that $\vec{F}(x,y) = \nabla f(x,y)$.

   (b) Find a potential function $f(x,y)$ for $\vec{F}$.

   (c) Use the fundamental theorem of line integrals to evaluate $\int_C \vec{F}\cdot d\vec{r}$.

6. Let $\vec{F}(x,y) = 2x\,\hat{\imath} + 6y\,\hat{\jmath}$ and let C be the line segment from $(2,1)$ to $(5,4)$.

   (a) Verify that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Note that the partial derivatives the component functions of $\vec{F}$ are defined and continuous on the entire xy-plane. Thus $\vec{F}$ is conservative and so there exists a potential function $f(x,y)$ such that $\vec{F}(x,y) = \nabla f(x,y)$.

   (b) Find a potential function $f(x,y)$ for $\vec{F}$.

   (c) Use the fundamental theorem of line integrals to evaluate $\int_C \vec{F}\cdot d\vec{r}$.

7. Let $\vec{F}(x,y) = 2xy\,\hat{\imath} + x^2\,\hat{\jmath}$ and let C be the line segment from $(2,1)$ to $(5,4)$.

   (a) Verify that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Note that the partial derivatives the component functions of $\vec{F}$ are defined and continuous on the entire xy-plane. Thus $\vec{F}$ is conservative and so there exists a potential function $f(x,y)$ such that $\vec{F}(x,y) = \nabla f(x,y)$.

   (b) Find a potential function $f(x,y)$ for $\vec{F}$.

   (c) Use the fundamental theorem of line integrals to evaluate $\int_C \vec{F}\cdot d\vec{r}$.

8. Let C be the part of the parabola $y = 3x^2$ from the point $(1,3)$ to $(2,12)$. Let $\vec{F} = \langle 2, 1\rangle$ be a vector field. Calculate $\int_C \vec{F}\cdot\vec{T}\,ds$ in the following two ways:

   (a) Parameterize the curve C and use the formula $\int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}\cdot\vec{r}\,'\,dt$.

   (b) Calculate $\int_C \vec{F}\cdot\vec{T}\,ds$ using the fundamental theorem of line integrals.

9. Let C be the part of the cubic $y = 2x^3$ from the point $(1,2)$ to $(4,128)$. Let $\vec{F} = \langle 2x, 2y\rangle$ be a vector field. Calculate $\int_C \vec{F}\cdot\vec{T}\,ds$ in the following two ways:

   (a) Parameterize the curve C and use the formula $\int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}\cdot\vec{r}\,'\,dt$.

   (b) Calculate $\int_C \vec{F}\cdot\vec{T}\,ds$ using the fundamental theorem of line integrals.

10. Use Green's theorem to evaluate the line integral.

   (a) $\int_C ye^x\, dx + 2e^x\, dy$, where C is the rectangle with vertices $(0,0)$, $(3,0)$, $(3,4)$, and $(0,4)$.

   (b) $\oint_C (x^2+y^2)\, dx + (x^2-y^2)\, dy$, where C is the triangle with vertices $(0,0)$, $(2,1)$, and $(0,1)$.

   (c) $\int_C y^3\, dx - x^3\, dy$, where C is the circle $x^2+y^2=4$.

11. Use Green's theorem to evaluate $\oint_C \vec{F}\cdot d\vec{r}$, where $\vec{F}(x,y) = \langle y\cos x - xy\sin x, xy + x\cos x\rangle$ and C is the triangle from $(0,0)$ to $(0,4)$ to $(2,0)$ to $(0,0)$.

12. Let $\vec{F} = xy\,\hat{\imath} + x\,\hat{\jmath}$ and let C be the closed curve $x^2+y^2=49$. If D is the disk enclosed by the curve C, use Green's theorem to evaluate $\oint_C \vec{F}\cdot d\vec{r}$.

13. If $\vec{F} = xyz\,\hat{\imath} + yz\,\hat{\jmath} + xy\,\hat{\boldsymbol{k}}$, find curl $\vec{F} = \nabla\times\vec{F}$.

14. If $\vec{F} = x^2y\,\hat{\imath} + z\,\hat{\jmath} + z^3\,\hat{\boldsymbol{k}}$, find div $\vec{F} = \nabla\cdot\vec{F}$.

15. Identify the surface with vector equation $\vec{r}(u,v) = \langle 3\cos v, u, \sin v\rangle$, $-1\le u\le 1$.

16. Find a parametric representation for the given surface.

   (a) The plane through the origin that contains the vectors $\hat{\imath}-\hat{\jmath}$ and $\hat{\jmath}-\hat{\boldsymbol{k}}$.

   (b) The plane that passes through the point $(0,-1,5)$ and contains the vectors $\langle 2,1,4\rangle$ and $\langle -3,2,5\rangle$.

   (c) The part of the sphere $x^2+y^2+z^2=4$ that lies above the cone $z=\sqrt{x^2+y^2}$.

   (d) The part of the cylinder $x^2+z^2=9$ that lies above the xy-plane and between the planes $y=-4$ and $y=4$.

17. Find an equation of the tangent plane to the parametric surface given by $x=u+v$, $y=3u^2$, $z=u-v$ at the point $(2,3,0)$.

18. Find the area of the surface.

   (a) The part of the plane $3x+2y+z=6$ that lies in the first octant.

   (b) The part of the plane $x+2y+3z=1$ that lies inside the cylinder $x^2+y^2=3$.

   (c) The part of the surface $z=4-2x^2+y$ that lies above the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$.

   (d) The part of the surface $z=xy$ that lies within the cylinder $x^2+y^2=1$.

19. Evaluate the surface integral.

   (a) $\iint_S xz\, dS$, where S is the part of the plane $2x+2y+z=4$ that lies in the first octant.

   (b) $\iint_S x\, dS$, where S is the triangular region with vertices $(1,0,0)$, $(0,-2,0)$, and $(0,0,4)$.

   (c) $\iint_S y^2\, dS$, where S is the part of the sphere $x^2+y^2+z^2=1$ that lies above the cone $z=\sqrt{x^2+y^2}$.

   (d) $\iint_S (x^2z+y^2z)\, dS$, where S is the hemisphere $x^2+y^2+z^2=4$, $z\ge 0$.

20. Evaluate the surface integral (flux integral) $\iint_S \vec{F}\cdot d\vec{S}$ for the given vector field $\vec{F}$ and oriented surface S. If the surface is closed, use the positive (outward normal) orientation.

   (a) $\vec{F}(x,y,z) = xy\,\hat{\imath} + yz\,\hat{\jmath} + zx\,\hat{\boldsymbol{k}}$, S is the part of the paraboloid $z=4-x^2-y^2$ that lies above the square $0\le x\le 1$, $0\le y\le 1$ and has upward orientation.

   (b) $\vec{F}(x,y,z) = y\,\hat{\imath} + y\,\hat{\jmath} + z^2\,\hat{\boldsymbol{k}}$, S is the sphere with radius 1 and center the origin.

   (c) $\vec{F}(x,y,z) = y\,\hat{\imath} - x\,\hat{\jmath} + 2z\,\hat{\boldsymbol{k}}$, S is the hemisphere $x^2+y^2+z^2=4$, $z\ge 0$, oriented downward.

21. Use Stokes' theorem to evaluate $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$, where $\vec{F}(x,y,z) = ze^y\,\hat{\imath} + x\cos y\,\hat{\jmath} + xz\sin y\,\hat{\boldsymbol{k}}$ and S is the hemisphere $x^2+y^2+z^2=16$, $y\ge 0$, oriented in the direction of the positive y-axis.

22. Use Stokes' theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y, z) = \langle x + y^2, y + z^2, z + x^2 \rangle$ and C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, oriented counterclockwise as view from above.

23. Use Stokes' theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y, z) = \langle xy, yz, zx \rangle$ and C is the boundary of the paraboloid $z = 1 - x^2 - y^2$ in the first octant, oriented counterclockwise as view from above.

24. Use the Divergence theorem to calculate the surface integral $\iint\limits_S \vec{F} \cdot d\vec{S}$, i.e., calculate the flux of $\vec{F}$ across S.

    (a) $\vec{F}(x, y, z) = xye^z\ \hat{\mathbf{\imath}} + xy^2z^3\ \hat{\mathbf{\jmath}} - ye^z\ \hat{\boldsymbol{k}}$, S is the surface of the box bounded by the coordinate planes and the planes $x = 3$, $y = 2$ and $z = 1$

    (b) $\vec{F}(x, y, z) = \langle x^3 + y^3, y^3 + z^3, z^3 + x^3 \rangle$, S is the sphere of radius 2 centered at the origin

    (c) $\vec{F}(x, y, z) = (2x^3 + y^3)\ \hat{\mathbf{\imath}} + (y^3 + z^3)\ \hat{\mathbf{\jmath}} + 3y^2z\ \hat{\boldsymbol{k}}$, S is the surface of the solid bounded by the paraboloid $z = 1 - x^2 - y^2$ and the xy-plane.