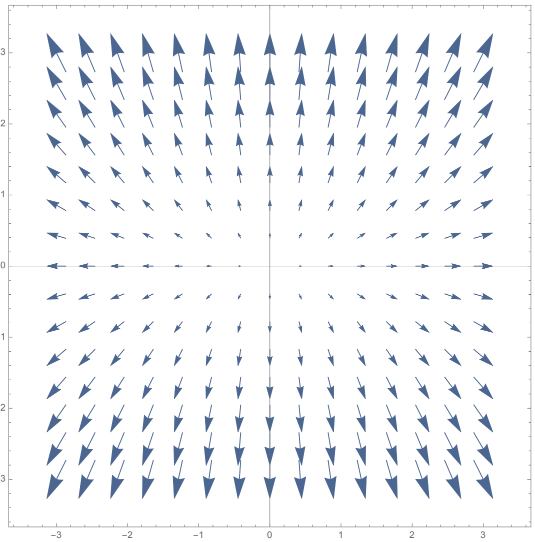
Final Exam Review Sheet Solutions

MATH 283

1. Sketch the vector field.

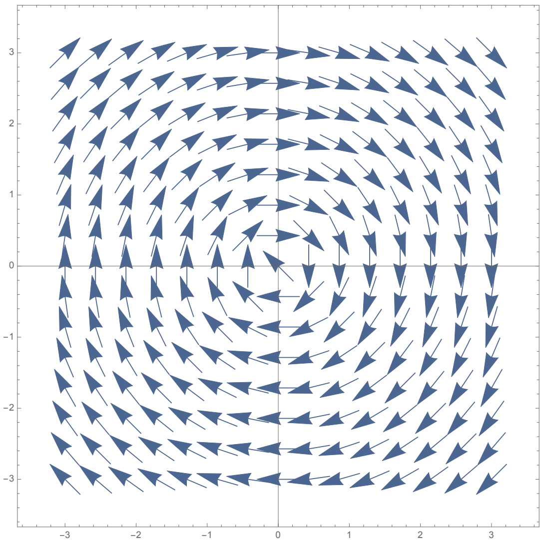
   (a) $\vec{F}(x,y) = \frac{1}{2}x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$

   Solution:



   (b) $\vec{F}(x,y) = \dfrac{y\,\hat{\mathbf{i}} - x\,\hat{\mathbf{j}}}{\sqrt{x^2+y^2}}$

   Solution:



2. Evaluate the line integral.

   (a) $\displaystyle\int_C xy^4\;ds$, where C is the right half of the circle $x^2+y^2=16$.

   Solution: Parameterize C via $\vec{r}(t) = \langle 4\cos t, 4\sin t\rangle$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. Then $\vec{r}\,'(t) = \langle -4\sin t, 4\cos t\rangle$ and so

$$\begin{aligned}\int_C xy^4\;ds &= \int_{-\pi/2}^{\pi/2} f(\vec{r}(t))\;|\vec{r}\,'(t)|\;dt \\ &= \int_{-\pi/2}^{\pi/2} (4\cos t)(4\sin t)^4\sqrt{(-4\sin t)^2 + (4\cos t)^2}\;dt \\ &= \int_{-\pi/2}^{\pi/2} 4096\cos t\sin^4 t\;dt \\ &= \frac{8192}{5}\end{aligned}$$

   (b) $\displaystyle\int_C xe^y\;ds$, where C is the line segment from $(2,0)$, to $(5,4)$.

   Solution: Parameterize C via $\vec{r}(t) = \langle 2+3t, 4t\rangle$, $0 \le t \le 1$. Then $\vec{r}\,'(t) = \langle 3,4\rangle$ and so

$$\begin{aligned}\int_C xe^y\ ds &= \int_0^1 f(\vec{r}(t))\ |\vec{r}\,'(t)|\ dt\\ &= \int_0^1 (2+3t)e^{4t}\sqrt{3^2+4^2}\ dt\\ &= \int_0^1 5(2+3t)e^{4t}\ dt\\ &= \int_0^1 \left(10e^{4t}+15te^{4t}\right)\ dt\\ &= \frac{85e^4-25}{16}\end{aligned}$$

(c) $\int_C y^2z\ ds$, where C is the line segment from $(3,1,2)$ to $(1,2,5)$.
Solution: Paramterize C via $\vec{r}(t)=\langle 3-2t, 1+t, 2+3t\rangle$, $0\le t\le 1$. Then $\vec{r}\,'(t)=\langle -2,1,3\rangle$ and so

$$\begin{aligned}\int_C y^2z\ ds &= \int_0^1 f(\vec{r}(t))\ |\vec{r}\,'(t)|\ dt\\ &= \int_0^1 (1+t)^2(2+3t)\sqrt{(-2)^2+1^2+3^2}\ dt\\ &= \int_0^1 (3t^3+8t^2+7t+2)\sqrt{14}\ dt\\ &= \frac{107\sqrt{14}}{12}\end{aligned}$$

(d) $\int_C x^2\ dx+y^2\ dy$, where C consists of the arc of the circle $x^2+y^2=4$ from $(2,0)$ to $(0,2)$ followed by the line segment from $(0,2)$ to $(4,3)$.
Solution: Since C is made up of two pieces, say C_1 and C_2, we parameterize C_1 via $\vec{r}_1(t)=\langle 2\cos t, 2\sin t\rangle$, $0\le t\le \frac{\pi}{2}$ and parameterize C_2 via $\vec{r}_2(t)=\langle 4t, 2+t\rangle$, $0\le t\le 1$. Then

$$\int_C x^2\ dx+y^2\ dy = \int_{C_1} x^2\ dx+y^2\ dy + \int_{C_2} x^2\ dx+y^2\ dy.$$

We will calculate each piece separately. First, on C_1 we have

$$\begin{aligned}\int_{C_1} x^2\ dx+y^2\ dy &= \int_{C_1} x^2\ dx + \int_{C_1} y^2\ dy\\ &= \int_0^{\pi/2} (x(t))^2x'(t)\ dt + \int_0^{\pi/2} (y(t))^2y'(t)\ dt\\ &= \int_0^{\pi/2} (2\cos t)^2(-2\sin t)\ dt + \int_0^{\pi/2} (2\sin t)^2(2\cos t)\ dt\\ &= -\frac{8}{3}+\frac{8}{3}=0.\end{aligned}$$

On C_2 we have

$$\begin{aligned}\int_{C_2} x^2\ dx+y^2\ dy &= \int_{C_2} x^2\ dx + \int_{C_2} y^2\ dy\\ &= \int_0^1 (x(t))^2x'(t)\ dt + \int_0^1 (y(t))^2y'(t)\ dt\\ &= \int_0^1 (4t)^2(4)\ dt + \int_0^1 (2+t)^2(1)\ dt\\ &= \int_0^1 (65t^2+4t+4)\ dt\\ &= \frac{83}{3}.\end{aligned}$$

Thus, $\displaystyle\int_C x^2\ dx + y^2\ dy = \int_{C_1} x^2\ dx + y^2\ dy + \int_{C_2} x^2\ dx + y^2\ dy = 0 + \frac{83}{3} = \frac{83}{3}$.

3. Evaluate $\displaystyle\int_C \vec{F}\cdot d\vec{r}$, where C is the curve given by the vector function $\vec{r}(t)$.

   (a) $\vec{F}(x,y) = xy^2\ \hat{\mathbf{\imath}} - x^2\ \hat{\mathbf{\jmath}}$;
   $\vec{r}(t) = t^3\ \hat{\mathbf{\imath}} + t^2\ \hat{\mathbf{\jmath}},\quad 0 \le t \le 1.$
   Solution:

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_0^1 \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\ dt\\ &= \int_0^1 \left\langle t^3(t^2)^2, -(t^3)^2\right\rangle\cdot\left\langle 3t^2, 2t\right\rangle\ dt\\ &= \int_0^1 (3t^9 - 2t^7)\ dt\\ &= \frac{1}{20}.\end{aligned}$$

   (b) $\vec{F}(x,y,z) = x\ \hat{\mathbf{\imath}} + y\ \hat{\mathbf{\jmath}} + xy\ \hat{\boldsymbol{k}}$;
   $\vec{r}(t) = \cos t\ \hat{\mathbf{\imath}} + \sin t\ \hat{\mathbf{\jmath}} + t\ \hat{\boldsymbol{k}},\quad 0 \le t \le \pi.$
   Solution:

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_0^\pi \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\ dt\\ &= \int_0^\pi \langle\cos t, \sin t, \cos t\sin t\rangle\cdot\langle-\sin t, \cos t, 1\rangle\ dt\\ &= \int_0^\pi (-\cos t\sin t + \sin t\cos t + \cos t\sin t)\ dt\\ &= \int_0^\pi \cos t\sin t\ dt\\ &= 0.\end{aligned}$$

4. Let $\vec{F}(x,y) = -y\ \hat{\mathbf{\imath}} + x\ \hat{\mathbf{\jmath}}$ and let C be the line segment from $(2,1)$ to $(5,4)$.
   Note: $\vec{F}$ is not conservative since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ and so you cannot use the fundamental theorem of line integrals to solve parts (b) and (d).

   (a) Find a parameterization $\vec{r}(t) = \langle x(t), y(t)\rangle$ for the line segment C such that $0 \le t \le 1$.
   Solution: $\vec{r}(t) = \langle 2+3t, 1+3t\rangle$, $0 \le t \le 1$

   (b) Evaluate $\displaystyle\int_C \vec{F}\cdot d\vec{r}$ using the parameterization given in (a).
   Solution:

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_0^1 \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\ dt\\ &= \int_0^1 \langle-(1+3t), 2+3t\rangle\cdot\langle 3, 3\rangle\ dt\\ &= \int_0^1 3\ dt\\ &= 3.\end{aligned}$$

   (c) Find a parameterization $\vec{r}(t) = \langle x(t), y(t)\rangle$ for the line segment C such that $0 \le t \le \frac{1}{2}$.
   Solution: $\vec{r}(t) = \langle 2+6t, 1+6t\rangle$, $0 \le t \le \frac{1}{2}$

   (d) Evaluate $\displaystyle\int_C \vec{F}\cdot d\vec{r}$ using the parameterization given in (c).

Solution:

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_0^{1/2} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\ dt\\ &= \int_0^{1/2} \langle -(1+6t), 2+6t\rangle \cdot \langle 6,6\rangle\ \ dt\\ &= \int_0^{1/2} 6\ dt\\ &= 3.\end{aligned}$$

5. Let $\vec{F}(x,y) = y\ \hat{\imath} + x\ \hat{\jmath}$ and let C be the line segment from $(2,1)$ to $(5,4)$.

   (a) Verify that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Note that the partial derivatives the component functions of $\vec{F}$ are defined and continuous on the entire xy-plane. Thus $\vec{F}$ is conservative and so there exists a potential function $f(x,y)$ such that $\vec{F}(x,y) = \nabla f(x,y)$.
   Solution: $\frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x}$

   (b) Find a potential function $f(x,y)$ for $\vec{F}$.
   Solution: $f(x,y) = xy$

   (c) Use the fundamental theorem of line integrals to evaluate $\displaystyle\int_C \vec{F}\cdot d\vec{r}$.

   Solution: $\displaystyle\int_C \vec{F}\cdot d\vec{r} = f(5,4) - f(2,1) = (5)(4) - (2)(1) = 18.$

6. Let $\vec{F}(x,y) = 2x\ \hat{\imath} + 6y\ \hat{\jmath}$ and let C be the line segment from $(2,1)$ to $(5,4)$.

   (a) Verify that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Note that the partial derivatives the component functions of $\vec{F}$ are defined and continuous on the entire xy-plane. Thus $\vec{F}$ is conservative and so there exists a potential function $f(x,y)$ such that $\vec{F}(x,y) = \nabla f(x,y)$.
   Solution: $\frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x}$

   (b) Find a potential function $f(x,y)$ for $\vec{F}$.
   Solution: $f(x,y) = x^2 + 3y^2$

   (c) Use the fundamental theorem of line integrals to evaluate $\displaystyle\int_C \vec{F}\cdot d\vec{r}$.

   Solution: $\displaystyle\int_C \vec{F}\cdot d\vec{r} = f(5,4) - f(2,1) = (5^2 + 3(4)^2) - (2^2 + 3(1)^2) = 66.$

7. Let $\vec{F}(x,y) = 2xy\ \hat{\imath} + x^2\ \hat{\jmath}$ and let C be the line segment from $(2,1)$ to $(5,4)$.

   (a) Verify that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Note that the partial derivatives the component functions of $\vec{F}$ are defined and continuous on the entire xy-plane. Thus $\vec{F}$ is conservative and so there exists a potential function $f(x,y)$ such that $\vec{F}(x,y) = \nabla f(x,y)$.
   Solution: $\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$

   (b) Find a potential function $f(x,y)$ for $\vec{F}$.
   Solution: $f(x,y) = x^2 y$

   (c) Use the fundamental theorem of line integrals to evaluate $\displaystyle\int_C \vec{F}\cdot d\vec{r}$.

   Solution: $\displaystyle\int_C \vec{F}\cdot d\vec{r} = f(5,4) - f(2,1) = 5^2(4) - 2^2(1) = 96.$

8. Let C be the part of the parabola $y = 3x^2$ from the point $(1,3)$ to $(2,12)$. Let $\vec{F} = \langle 2,1\rangle$ be a vector field. Calculate $\displaystyle\int_C \vec{F}\cdot\vec{T}\ ds$ in the following two ways:

(a) Parameterize the curve C and use the formula $\displaystyle\int_C \vec{F}\cdot\vec{T}\ ds = \int_a^b \vec{F}\cdot\vec{r}\,'\ dt$.

Solution: $\vec{r}(t) = \left\langle t, 3t^2\right\rangle$, $1 \le t \le 2$ and so

$$\begin{aligned}\int_C \vec{F}\cdot\vec{T}\ ds &= \int_1^2 \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\ dt\\ &= \int_1^2 \langle 2,1\rangle\cdot\langle 1,6t\rangle\ dt\\ &= \int_1^2 (2+6t)\ dt\\ &= 11.\end{aligned}$$

(b) Calculate $\displaystyle\int_C \vec{F}\cdot\vec{T}\ ds$ using the fundamental theorem of line integrals.

Solution: Since $\frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x}$ and the partial derivatives of the component functions of $\vec{F}$ are defined and continuous everywhere, $\vec{F}$ is conservative. A potential function is given by $f(x,y) = 2x + y$ and so we have $\int_C \vec{F}\cdot\vec{T}\ ds = f(2,12) - f(1,3) = (2(2)+12) - (2(1)+3) = 16 - 5 = 11$.

9. Let C be the part of the cubic $y = 2x^3$ from the point $(1,2)$ to $(4,128)$. Let $\vec{F} = \langle 2x, 2y\rangle$ be a vector field. Calculate $\displaystyle\int_C \vec{F}\cdot\vec{T}\ ds$ in the following two ways:

(a) Parameterize the curve C and use the formula $\displaystyle\int_C \vec{F}\cdot\vec{T}\ ds = \int_a^b \vec{F}\cdot\vec{r}\,'\ dt$.

Solution: $\vec{r}(t) = \left\langle t, 2t^3\right\rangle$, $1 \le t \le 4$ and so

$$\begin{aligned}\int_C \vec{F}\cdot\vec{T}\ ds &= \int_1^4 \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\ dt\\ &= \int_1^4 \left\langle 2t, 2(2t^3)\right\rangle\cdot\left\langle 1,6t^2\right\rangle\ dt\\ &= \int_1^4 (2t+24t^5)\ dt\\ &= 16395.\end{aligned}$$

(b) Calculate $\displaystyle\int_C \vec{F}\cdot\vec{T}\ ds$ using the fundamental theorem of line integrals.

Solution: Since $\frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x}$ and the partial derivatives of the component functions of $\vec{F}$ are defined and continuous everywhere, $\vec{F}$ is conservative. A potential function is given by $f(x,y) = x^2 + y^2$ and so we have $\int_C \vec{F}\cdot\vec{T}\ ds = f(4,128) - f(1,2) = (4^2 + 128^2) - (1^2 + 2^2) = 16395$.

10. Use Green's theorem to evaluate the line integral.

(a) $\displaystyle\int_C ye^x\ dx + 2e^x\ dy$, where C is the rectangle with vertices $(0,0)$, $(3,0)$, $(3,4)$, and $(0,4)$.

Solution:

$$\begin{aligned}\int_C ye^x\ dx + 2e^x\ dy &= \iint_D \left(\frac{\partial}{\partial x}(2e^x) - \frac{\partial}{\partial y}(ye^x)\right)\ dA\\ &= \iint_D e^x\ dA\\ &= \int_0^4\int_0^3 e^x\ dx\ dy\\ &= 4e^3 - 4.\end{aligned}$$

(b) $\oint_C (x^2+y^2)\,dx + (x^2-y^2)\,dy$, where C is the triangle with vertices $(0,0)$, $(2,1)$, and $(0,1)$.

Solution:

$$\begin{aligned}\oint_C (x^2+y^2)\,dx + (x^2-y^2)\,dy &= \iint_D \left(\frac{\partial}{\partial x}(x^2-y^2) - \frac{\partial}{\partial y}(x^2+y^2)\right)\,dA\\ &= \iint\limits_D (2x-2y)\,dA\\ &= \int_0^2 \int_{x/2}^1 (2x-2y)\,dy\,dx\\ &= 0.\end{aligned}$$

(c) $\int_C y^3\,dx - x^3\,dy$, where C is the circle $x^2+y^2=4$.

Solution:

$$\begin{aligned}\int_C y^3\,dx - x^3\,dy &= \iint\limits_D \left(\frac{\partial}{\partial x}(-x^3) - \frac{\partial}{\partial y}(y^3)\right)\,dA\\ &= \iint\limits_D (-3x^2-3y^2)\,dA.\end{aligned}$$

It is now easiest to use polar coordinates to evaluate this double integral:

$$\begin{aligned}\iint\limits_D (-3x^2-3y^2)\,dA &= \int_0^{2\pi}\int_0^2 (-3r^2)r\,dr\,d\theta\\ &= -24\pi.\end{aligned}$$

11. Use Green's theorem to evaluate $\oint_C \vec{F}\cdot d\vec{r}$, where $\vec{F}(x,y) = \langle y\cos x - xy\sin x, xy + x\cos x\rangle$ and C is the triangle from $(0,0)$ to $(0,4)$ to $(2,0)$ to $(0,0)$.

Solution: Observe that as written, the curve C is oriented clockwise, so to apply Green's theorem we will need to reorient the curve in the positive direction (and in the process, we'll pick up a negative sign). We have

$$\begin{aligned}\oint_C \vec{F}\cdot d\vec{r} &= -\int_{-C}\vec{F}\cdot d\vec{r}\\ &= -\iint\limits_D \left(\frac{\partial}{\partial x}(xy + x\cos x) - \frac{\partial}{\partial y}(y\cos x - xy\sin x)\right)\,dA\\ &= -\iint\limits_D y\,dA\\ &= -\int_0^2\int_0^{4-2x} y\,dy\,dx\\ &= -\frac{16}{3}.\end{aligned}$$

12. Let $\vec{F} = xy\,\hat{\imath} + x\,\hat{\jmath}$ and let C be the closed curve $x^2+y^2=49$. If D is the disk enclosed by the curve C, use Green's theorem to evaluate $\oint_C \vec{F}\cdot d\vec{r}$.

Solution:

$$\begin{aligned}\oint_C \vec{F}\cdot d\vec{r} &= \iint\limits_D \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(xy)\right)\,dA\\ &= \iint\limits_D (1-x)\,dA.\end{aligned}$$

Again, it is easiest to evaluate this double integral in polar coordinates.

$$\iint_D (1-x)\ dA = \int_0^{2\pi}\int_0^7 (1 - r\cos\theta) r\ dr\ d\theta$$
$$= 49\pi.$$

13. If $\vec{F} = xyz\ \hat{\mathbf{\imath}} + yz\ \hat{\mathbf{\jmath}} + xy\ \hat{\boldsymbol{k}}$, find curl $\vec{F} = \nabla \times \vec{F}$.

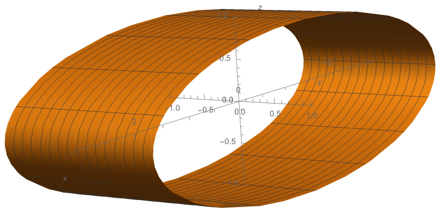
    Solution: curl $\vec{F} = (x-y)\ \hat{\mathbf{\imath}} + (xy - y)\ \hat{\mathbf{\jmath}} - xz\ \hat{\boldsymbol{k}}$

14. If $\vec{F} = x^2 y\ \hat{\mathbf{\imath}} + z\ \hat{\mathbf{\jmath}} + z^3\ \hat{\boldsymbol{k}}$, find div $\vec{F} = \nabla \cdot \vec{F}$.

    Solution: div $\vec{F} = 2xy + 3z^2$

15. Identify the surface with vector equation $\vec{r}(u,v) = \langle 3\cos v, u, \sin v\rangle$, $-1 \le u \le 1$.

    Solution: This is a portion of an elliptic cylinder with axis the y-axis.



16. Find a parametric representation for the given surface.

    (a) The plane through the origin that contains the vectors $\hat{\mathbf{\imath}} - \hat{\mathbf{\jmath}}$ and $\hat{\mathbf{\jmath}} - \hat{\boldsymbol{k}}$.

    Solution: A parametric equation for a plane through a point with position vector $\vec{r}_0$ and containing the vectors $\vec{a}$ and $\vec{b}$ has the form $\vec{r}(u,v) = \vec{r}_0 + u\vec{a} + v\vec{b}$. We have $\vec{a} = \hat{\mathbf{\imath}} - \hat{\mathbf{\jmath}}$ and $\vec{b} = \hat{\mathbf{\jmath}} - \hat{\boldsymbol{k}}$ and $\vec{r}_0 = \langle 0,0,0\rangle$. Thus

    $$\vec{r}(u,v) = \langle 0,0,0\rangle + u(\hat{\mathbf{\imath}} - \hat{\mathbf{\jmath}}) + v(\hat{\mathbf{\jmath}} - \hat{\boldsymbol{k}}) = u\ \hat{\mathbf{\imath}} + (v-u)\ \hat{\mathbf{\jmath}} - v\ \hat{\boldsymbol{k}}.$$

    (b) The plane that passes through the point $(0,-1,5)$ and contains the vectors $\langle 2,1,4\rangle$ and $\langle -3,2,5\rangle$.

    Solution: Similar to the previous question, we have

    $$\vec{r}(u,v) = \langle 0,-1,5\rangle + u\,\langle 2,1,4\rangle + v\,\langle -3,2,5\rangle = \langle 2u - 3v, -1 + u + 2v, 5 + 4u + 5v\rangle$$

    .

    (c) The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$.

    Solution: We use the spherical angles θ, ϕ as parameters. The radius of the sphere is $\rho = 2$. Thus we have

    $$x(\phi,\theta) = 2\sin\phi\cos\theta$$
    $$y(\phi,\theta) = 2\sin\phi\sin\theta$$
    $$z(\phi,\theta) = 2\cos\phi.$$

    We now need to restrict the parameters to get the desired portion of the sphere. The portion of the sphere above the cone is given by $0 \le \phi \le \frac{\pi}{4}$, and there is no restriction on θ, so $0 \le \theta \le 2\pi$.

    (d) The part of the cylinder $x^2 + z^2 = 9$ that lies above the xy-plane and between the planes $y = -4$ and $y = 4$.

    Solution: This cylinder has axis the y-axis. Thus we have

    $$\vec{r}(s,t) = \langle 3\cos t, s, 3\sin t\rangle\,, \quad -4 \le s \le 4,\ 0 \le t \le \pi.$$

17. Find an equation of the tangent plane to the parametric surface given by $x = u + v$, $y = 3u^2$, $z = u - v$ at the point $(2,3,0)$.

    Solution: We have $\vec{r}_u = \langle 1, 6u, 1\rangle$ and $\vec{r}_v = \langle 1, 0, -1\rangle$. At the point $(2,3,0)$ we solve for the corresponding parameter values, e.g., since $3 = y = 3u^2$, we have $u = \pm 1$. If $u = 1$, then $v = 1$ since $2 = x = u + v$, which does indeed give $z = u - v = 0$ (the case that $u = -1$ has no solutions).

    $$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\mathbf{\imath}} & \hat{\mathbf{\jmath}} & \hat{\boldsymbol{k}} \\ 1 & 6 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \langle -6, 2, -6\rangle\,.$$

This is a normal vector to the tangent plane, so an equation for the tangent plane is given by $-6(x-2)+2(y-3)-6(z-0)=0$, or equivalently, $3x-y+3z=3$.

18. Find the area of the surface.

    (a) The part of the plane $3x+2y+z=6$ that lies in the first octant.
    Solution: We can write the plane as $z=6-3x-2y$ and so the surface area is given by

$$
\begin{aligned}
A(S) &= \iint\limits_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = \iint\limits_D \sqrt{1+(-3)^2+(-2)^2}\,dA \\
&= \iint\limits_D \sqrt{14}\,dA = \int_0^2 \int_0^{3-\frac{3}{2}x} \sqrt{14}\,dy\,dx = 3\sqrt{14}.
\end{aligned}
$$

    (b) The part of the plane $x+2y+3z=1$ that lies inside the cylinder $x^2+y^2=3$.
    Solution: Again we have the plane $z=\frac{1}{3}-\frac{1}{3}x-\frac{2}{3}y$ and so

$$
\begin{aligned}
A(S) &= \iint\limits_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = \iint\limits_D \sqrt{1+\left(-\frac{1}{3}\right)^2+\left(-\frac{2}{3}\right)^2}\,dA \\
&= \iint\limits_D \sqrt{\frac{14}{9}}\,dA = \frac{\sqrt{14}}{3}\iint\limits_D dA = \frac{\sqrt{14}}{3}\int_0^{2\pi}\int_0^{\sqrt{3}} r\,dr\,d\theta = \sqrt{14}\pi.
\end{aligned}
$$

    (c) The part of the surface $z=4-2x^2+y$ that lies above the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$.
    Solution:

$$
\begin{aligned}
A(S) &= \iint\limits_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = \iint\limits_D \sqrt{1+(-4x)^2+(1)^2}\,dA \\
&= \iint\limits_D \sqrt{2+16x^2}\,dA = \int_0^1\int_0^x \sqrt{2+16x^2}\,dy\,dx = \frac{13\sqrt{2}}{12}.
\end{aligned}
$$

    (d) The part of the surface $z=xy$ that lies within the cylinder $x^2+y^2=1$.
    Solution:

$$
\begin{aligned}
A(S) &= \iint\limits_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = \iint\limits_D \sqrt{1+y^2+x^2}\,dA \\
&= \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}\,r\,dr\,d\theta = \frac{2\pi}{3}(2\sqrt{2}-1).
\end{aligned}
$$

19. Evaluate the surface integral.

    (a) $\iint\limits_S xz\,dS$, where S is the part of the plane $2x+2y+z=4$ that lies in the first octant.
    Solution: We parameterize S as $\vec{r}(x,y)=\langle x,y,4-2x-2y\rangle$ and so we have

$$
\begin{aligned}
\iint\limits_S xz\,dS &= \iint\limits_D x(4-2x-2y)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA \\
&= \iint\limits_D (4x-2x^2-2xy)\sqrt{1+(-2)^2+(-2)^2}\,dA \\
&= \int_0^2\int_0^{2-x} 3(4x-2x^2-2xy)\,dy\,dx = 4.
\end{aligned}
$$

(b) $\iint\limits_S x\ dS$, where S is the triangular region with vertices $(1,0,0)$, $(0,-2,0)$, and $(0,0,4)$.

Solution: The region S is part of a plane, so in order to parameterize it, we first need an equation of the plane. Denote the vertices above as P, Q, R, respectively. Then $\vec{PQ} = \langle -1,-2,0\rangle$ and $\vec{PR} = \langle -1,0,4\rangle$. Thus vector normal to the plane is

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{\mathbf{\imath}} & \hat{\mathbf{\jmath}} & \hat{\boldsymbol{k}} \\ -1 & -2 & 0 \\ -1 & 0 & 4 \end{vmatrix} = \langle -8, 4, -2\rangle .$$

We'll scale this normal vector to $\langle -4,2,-1\rangle$ (just to make it a little easier to work with). Thus an equation for the plane is $-4(x-1)+2(y-0)-(z-0)=0$, or equivalently, $z = 4-4x+2y$. So a parameterization of the plane is $\vec{r}(x,y) = \langle x, y, 4-4x+2y\rangle$. Then

$$\begin{aligned} \iint\limits_S x\ dS &= \iint\limits_D x\sqrt{1+(-4)^2+(2)^2}\ dA = \iint\limits_D x\sqrt{21} \\ &= \int_0^1 \int_{2x-2}^0 x\sqrt{21}\ dy\ dx = \frac{\sqrt{21}}{3}. \end{aligned}$$

(c) $\iint\limits_S y^2\ dS$, where S is the part of the sphere $x^2+y^2+z^2=1$ that lies above the cone $z=\sqrt{x^2+y^2}$.

Solution: We parameterize S using the spherical parameters $\vec{r}(\phi,\theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$. The portion of the sphere that lies above the cone is given by restricting the parameters to $0 \le \phi \le \frac{\pi}{4}$, $0 \le \theta \le 2\pi$. Then we have

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \hat{\mathbf{\imath}} & \hat{\mathbf{\jmath}} & \hat{\boldsymbol{k}} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix} = \left\langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi\right\rangle .$$

Hence $|\vec{r}_\phi \times \vec{r}_\theta| = \sin\phi$. Finally, we have

$$\begin{aligned} \iint\limits_S y^2\ dS &= \iint\limits_D y^2\ |\vec{r}_\phi \times \vec{r}_\theta|\ dA \\ &= \iint\limits_D (\sin\phi\sin\theta)^2 \cdot \sin\phi\ dA \\ &= \iint\limits_D \sin^3\phi\sin^2\theta\ dA \\ &= \int_0^{\pi/4}\int_0^{2\pi} \sin^3\phi\sin^2\theta\ d\theta\ d\phi \\ &= \left(\frac{\sqrt{2}}{2} - \frac{(\sqrt{2})^3}{24} - \frac{2}{3}\right)\pi \\ &= \left(\frac{2}{3} - \frac{5\sqrt{2}}{12}\right)\pi. \end{aligned}$$

(d) $\iint\limits_S (x^2z+y^2z)\ dS$, where S is the hemisphere $x^2+y^2+z^2=4$, $z \ge 0$.

Solution: Using the spherical parameterization, we have $\vec{r}(\phi,\theta) = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi\rangle$, with $0 \le \phi \le \frac{\pi}{2}$, $0 \le \theta \le 2\pi$. Then $\vec{r}_\phi \times \vec{r}_\theta = \left\langle 4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\sin\phi\cos\phi\right\rangle$ and so $|\vec{r}_\phi \times \vec{r}_\theta| = 4\sin\phi$.

Thus

$$\begin{aligned}
\iint\limits_S (x^2z + y^2z)\ dS &= \iint\limits_D (x^2z+y^2z)|\vec{r}_\phi \times \vec{r}_\theta|\ dA \\
&= \iint\limits_D ((2\sin\phi\cos\theta)^2(2\cos\phi) + (2\sin\phi\sin\theta)^2(2\cos\phi)\cdot 4\sin\phi\ dA \\
&= \iint\limits_D 32\sin^3\phi\cos\phi\ dA \\
&= \int_0^{\pi/2}\int_0^{2\pi} 32\sin^3\phi\cos\phi\ d\theta\ d\phi = 16\pi.
\end{aligned}$$

20. Evaluate the surface integral (flux integral) $\iint\limits_S \vec{F}\cdot d\vec{S}$ for the given vector field $\vec{F}$ and oriented surface S. If the surface is closed, use the positive (outward normal) orientation.

(a) $\vec{F}(x,y,z) = xy\ \hat{\mathbf{\imath}} + yz\ \hat{\mathbf{\jmath}} + zx\ \hat{\boldsymbol{k}}$, S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \le x \le 1$, $0 \le y \le 1$ and has upward orientation.
Solution: We parameterize S via $\vec{r}(x,y) = \left\langle x, y, 4-x^2-y^2\right\rangle$ and

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{\mathbf{\imath}} & \hat{\mathbf{\jmath}} & \hat{\boldsymbol{k}} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = \langle 2x, 2y, 1\rangle\,.$$

Thus

$$\begin{aligned}
\iint\limits_S \vec{F}\cdot d\vec{S} &= \iint\limits_D \vec{F}\cdot(\vec{r}_x\times\vec{r}_y)\ dA \\
&= \iint\limits_D \langle xy, yz, zx\rangle \cdot \langle 2x, 2y, 1\rangle\ dA \\
&= \iint\limits_D (2x^2y + 2y^2z + zx)\ dA.
\end{aligned}$$

We substitute $z = 4 - x^2 - y^2$ in the integrand and so we have

$$\begin{aligned}
\iint\limits_D (2x^2y + 2y^2(4-x^2-y^2) + (4-x^2-y^2)x)\ dA &= \int_0^1\int_0^1 (2x^2y + 8y^2 - 2x^2y^2 - 2y^4 + 4x - x^3 - xy^2)\ dy\ dx \\
&= \frac{713}{180}.
\end{aligned}$$

(b) $\vec{F}(x,y,z) = y\ \hat{\mathbf{\imath}} + y\ \hat{\mathbf{\jmath}} + z^2\ \hat{\boldsymbol{k}}$, S is the sphere with radius 1 and center the origin.
Solution: As usual, we parameterize S via $\vec{r}(\phi,\theta) = \langle\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$, with $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$. Then $\vec{r}_\phi \times \vec{r}_\theta = \left\langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi\right\rangle$. We now need to confirm that this normal vector gives the correct orientation on S (the sphere S is closed, so we need the positive (outward) normal). In the first octant, we have $0 \le \phi \le \frac{\pi}{2}$ and $0 \le \theta \le \frac{\pi}{2}$, and on these intervals, each component of $\vec{r}_\phi \times \vec{r}_\theta$ is positive, i.e., the normal vector is pointing outward from S. So this choice of normal vector does indeed give the correct orientation.

Thus

$$\begin{aligned}
\iint\limits_S \vec{F}\cdot d\vec{S} &= \iint\limits_D \vec{F}(\vec{r}(\phi,\theta))\cdot(\vec{r}_\phi\times\vec{r}_\theta)\ dA\\
&= \iint\limits_D \left\langle \sin\phi\sin\theta, \sin\phi\sin\theta, \cos^2\phi\right\rangle\cdot\left\langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi\right\rangle\ dA\\
&= \iint\limits_D (\sin^3\phi\sin\theta\cos\theta+\sin^3\phi\sin^2\theta+\sin\phi\cos^3\phi)\ dA\\
&= \int_0^\pi\int_0^{2\pi}(\sin^3\phi\sin\theta\cos\theta+\sin^3\phi\sin^2\theta+\sin\phi\cos^3\phi)\ d\theta\ d\phi\\
&= \frac{4\pi}{3}.
\end{aligned}$$

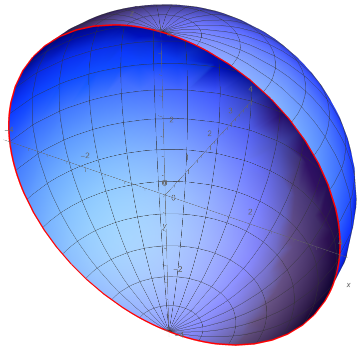
(c) $\vec{F}(x,y,z)=y\ \hat{\mathbf{\imath}}-x\ \hat{\mathbf{\jmath}}+2z\ \hat{\boldsymbol{k}}$, S is the hemisphere $x^2+y^2+z^2=4$, $z\geq 0$, oriented downward.

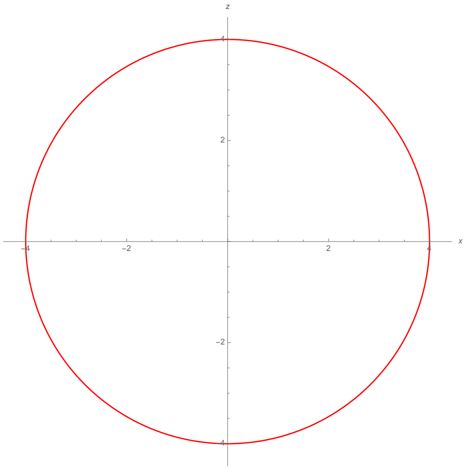
Solution: Our parameterization of S is $\vec{r}(\phi,\theta)=\langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi\rangle$, with $0\leq\phi\leq\frac{\pi}{2}$, $0\leq\theta\leq 2\pi$. We know (from part (b)) that the normal vector $\vec{r}_\phi\times\vec{r}_\theta$ gives the outward (in this case, upward) normal orientation, so to orient S downward, we need to take the negative of this: $-\vec{r}_\phi\times\vec{r}_\theta=\vec{r}_\theta\times\vec{r}_\phi=\left\langle -4\sin^2\phi\cos\theta, -4\sin^2\phi\sin\theta, -4\sin\phi\cos\phi\right\rangle$. Thus we have

$$\begin{aligned}
\iint\limits_S \vec{F}\cdot d\vec{S} &= \iint\limits_D \vec{F}(\vec{r}(\phi,\theta))\cdot(\vec{r}_\phi\times\vec{r}_\theta)\ dA\\
&= \iint\limits_D \langle 2\sin\phi\sin\theta, -2\sin\phi\cos\theta, 4\cos\phi\rangle\cdot\left\langle -4\sin^2\phi\cos\theta, -4\sin^2\phi\sin\theta, -4\sin\phi\cos\phi\right\rangle\ dA\\
&= \iint\limits_D \left(-16\sin\phi\cos^2\phi\right)\ dA\\
&= -16\int_0^{\pi/2}\int_0^{2\pi}\sin\phi\cos^2\phi\ d\theta\ d\phi\\
&= -\frac{32\pi}{3}.
\end{aligned}$$

21. Use Stokes' theorem to evaluate $\iint\limits_S \operatorname{curl}\ \vec{F}\cdot d\vec{S}$, where $\vec{F}(x,y,z)=ze^y\ \hat{\mathbf{\imath}}+x\cos y\ \hat{\mathbf{\jmath}}+xz\sin y\ \hat{\boldsymbol{k}}$ and S is the hemisphere $x^2+y^2+z^2=16$, $y\geq 0$, oriented in the direction of the positive y-axis.

Solution: We need a parameterization of the boundary curve C. The surface S is shown below, with the boundary curve C shown in red.





Here, the positive y-axis points into the page, and so the chosen normal orientation points that direction (outward from the surface of the hemisphere). The curve C lies in the xz-plane, and from our perspective we can see that the orientation of S induces the *clockwise* orientation of C. Thus we can parameterize C via $\vec{r}(t)=\langle 4\sin t, 0, 4\cos t\rangle$,

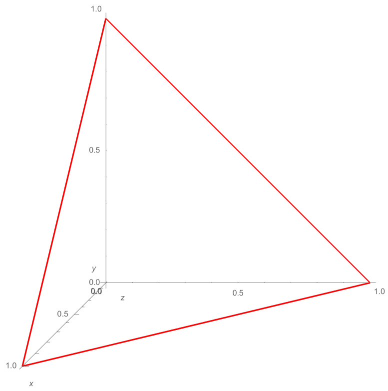
$0 \leq t \leq 2\pi$. Notice the difference in the components to how we usually parameterize circles; this is due to the clockwise orientation. As t increases from 0 to 2π, we actually start at the point $(x,z) = (0,1)$ and proceed around C in a clockwise manner (the usual parameterization, with our components switched, would have us start at the point $(x,z) = (1,0)$ and proceed around in a counterclockwise direction, which is backwards from what we need).

Finally, we can apply Stokes' theorem:

$$\begin{aligned}
\iint_S \text{curl } \vec{F} \cdot d\vec{S} &= \int_C \vec{F} \cdot d\vec{r} \\
&= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt \\
&= \int_0^{2\pi} \left\langle (4\cos t)e^0, 4\sin t \cos 0, (4\sin t)(4\cos t)\sin 0 \right\rangle \cdot \langle 4\cos t, 0, -4\sin t\rangle\ dt \\
&= \int_0^{2\pi} 16\cos^2 t\ dt \\
&= 16\int_0^{2\pi} \frac{1}{2}\left(1+\cos(2t)\right)\ dt \\
&= 8\left(t + \frac{1}{2}\sin(2t)\right)\Big|_0^{2\pi} \\
&= 16\pi.
\end{aligned}$$

22. Use Stokes' theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y,z) = \left\langle x+y^2, y+z^2, z+x^2\right\rangle$ and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, oriented counterclockwise as viewed from above.

Solution: The curve C is shown below, and is oriented counterclockwise from our perspective. The surface S bounded by C is part of the plane $x+y+z=1$.



If we parameterize the plane via $\vec{r}(x,y) = \langle x, y, 1-x-y\rangle$, then we have normal vector $\vec{r}_x \times \vec{r}_y = \langle 1,1,1\rangle$. We also have curl $\vec{F} = \nabla \times \vec{F} = \langle -2z, -2x, -2y\rangle$. Then by Stokes' theorem we have

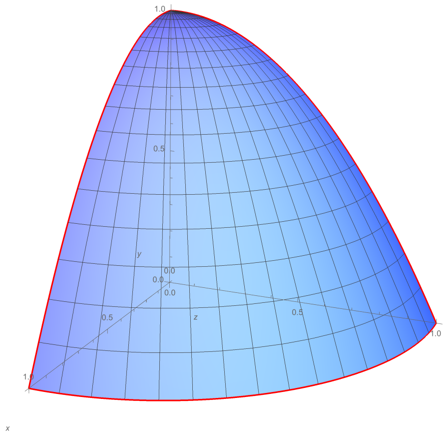
$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \iint_S \text{curl } \vec{F} \cdot d\vec{S} \\
&= \iint_D \text{curl } \vec{F}\left(\vec{r}(x,y)\right) \cdot \left(\vec{r}_x \times \vec{r}_y\right)\ dA \\
&= \iint_D \langle -2(1-x-y), -2x, -2y\rangle \cdot \langle 1,1,1\rangle\ dA \\
&= \iint_D (-2)\ dA.
\end{aligned}$$

The region D is the projection of S onto the xy-plane and so we have

$$= \iint\limits_D (-2)\, dA = -2 \int_0^1 \int_0^{1-x} dy\, dx = -1.$$

23. Use Stokes' theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y,z) = \langle xy, yz, zx \rangle$ and C is the boundary of the paraboloid $z = 1 - x^2 - y^2$ in the first octant, oriented counterclockwise as view from above.

    Solution: The surface S bounded by the curve C is the paraboloid $z = 1 - x^2 - y^2$, which we parameterize via $\vec{r}(s,t) = \left\langle s\cos t, s\sin t, 1 - s^2 \right\rangle$, $0 \le s \le 1$, $0 \le t \le \frac{\pi}{2}$ (we used polar coordinates here). The surface S and the curve C (in red) are shown below.



Then curl $\vec{F} = \langle -y, -z, -x \rangle$ and $\vec{r}_s \times \vec{r}_t = \left\langle 2s^2\cos t, 2s^2\sin t, s \right\rangle$. This normal vector does give the correct orientation for S (compatible with the given orientation of C). Thus by Stokes' theorem,

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \iint\limits_S \text{curl } \vec{F} \cdot d\vec{S} \\
&= \iint\limits_D \text{curl } \vec{F}\,(\vec{r}(s,t)) \cdot (\vec{r}_s \times \vec{r}_t)\; dA \\
&= \iint\limits_D \left\langle -s\sin t, s^2 - 1, -s\cos t \right\rangle \cdot \left\langle 2s^2\cos t, 2s^2\sin t, s \right\rangle\; dA \\
&= \iint\limits_D \left(-2s^3 \sin t\cos t + 2s^4\sin t - 2s^2\sin t - s^2\cos t \right)\; dA \\
&= \int_0^{\pi/2} \int_0^1 \left(-2s^3 \sin t\cos t + 2s^4\sin t - 2s^2\sin t - s^2\cos t \right)\; ds\; dt \\
&= -\frac{17}{20}.
\end{aligned}$$

24. Use the Divergence theorem to calculate the surface integral $\iint\limits_S \vec{F} \cdot d\vec{S}$, i.e., calculate the flux of $\vec{F}$ across S.

    (a) $\vec{F}(x,y,z) = xye^z\ \hat{\mathbf{\imath}} + xy^2z^3\ \hat{\mathbf{\jmath}} - ye^z\ \hat{\boldsymbol{k}}$, S is the surface of the box bounded by the coordinate planes and the planes $x = 3$, $y = 2$ and $z = 1$

    Solution: We have div $\vec{F} = \nabla \cdot \vec{F} = ye^z + 2xyz^3 - ye^z = 2xyz^3$. Then by the Divergence theorem, we have

$$\begin{aligned}
\iint\limits_S \vec{F} \cdot d\vec{S} &= \iiint\limits_B \text{div } \vec{F}\; dV \\
&= \int_0^3 \int_0^2 \int_0^1 2xyz^3\; dz\; dy\; dx \\
&= \frac{9}{2}.
\end{aligned}$$

(b) $\vec{F}(x,y,z) = \langle x^3+y^3, y^3+z^3, z^3+x^3 \rangle$, S is the sphere of radius 2 centered at the origin.

Solution: We have div $\vec{F} = \nabla \cdot \vec{F} = 3x^2+3y^2+3z^2$. Since S is a sphere, we will use spherical coordinates. Thus by the Divergence theorem, we have

$$\begin{aligned} \iint\limits_S \vec{F} \cdot d\vec{S} &= \iiint\limits_B \text{div } \vec{F} \; dV \\ &= \iiint\limits_B 3(x^2+y^2+z^2) \; dV \\ &= \int_0^{\pi} \int_0^{2\pi} \int_0^2 3\rho^2 \cdot \rho^2 \sin\phi \; d\rho \; d\theta \; d\phi \\ &= \frac{384\pi}{5}. \end{aligned}$$

(c) $\vec{F}(x,y,z) = (2x^3+y^3)\;\hat{\mathbf{i}} + (y^3+z^3)\;\hat{\mathbf{j}} + 3y^2z\;\hat{\boldsymbol{k}}$, S is the surface of the solid bounded by the paraboloid $z = 1-x^2-y^2$ and the xy-plane.

Solution: We have div $\vec{F} = \nabla \cdot \vec{F} = 6x^2+6y^2$. We will use cylindrical coordinate. Thus by the Divergence theorem, we have

$$\begin{aligned} \iint\limits_S \vec{F} \cdot d\vec{S} &= \iiint\limits_B \text{div } \vec{F} \; dV \\ &= \iiint\limits_B 6(x^2+y^2) \; dV \\ &= \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} 6r^2 \cdot r \; dz \; dr \; d\theta \\ &= \pi. \end{aligned}$$